Mandelbrot Set in a Non-Analytic Map

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A 2-dimensional non-analytic map is presented which generates a Mandelbrot set (as well as Julia sets) not different from those obtained of the complex-analytic logistic map itself.

For the complex logistic map:

$$z_{n+1} = z_n^2 + c \tag{1}$$

(with $z_n, c \in \mathbb{C}$), Metzler [1] recently found a connection between the Mandelbrot set [2] and a series of trachoid equations which bound the convergence area. Using the canonical transform $f(z) = (f_1(x, y), f_2(x, y))$, (1) can be considered as a function of two real variables,

$$x_{n+1} = x_n^2 - y_n^2 + c_1, \quad y_{n+1} = 2 x_n y_n + c_2 \tag{2}$$

(with $x_n, y_n, c_1, c_2 \in \mathbb{R}$), such that the Cauchy-Riemann differential equations

$$\partial f_1(x, y)/\partial x = \partial f_2(x, y)/\partial y, \tag{3a}$$

$$\partial f_1(x, y)/\partial y = -\partial f_2(x, y)/\partial x. \tag{3b}$$

are satisfied.

There are several studies on the effects of destroying analyticity in the complex-analytic logistic map by simply introducing a real perturbing term into (2) (see [3], [4]).

Here, a new 2-dimensional, non-analytic real map is presented, which also generates the Mandelbrot set:

$$x_{n+1} = 2 x_n y_n - y_n^2 + c_1, \quad y_{n+1} = 2 x_n y_n - x_n^2 + c_2 \tag{4}$$

(with x_n, y_n, c_1, $c_2 \in \mathbb{R}$). This map generates a slightly stretched Mandelbrot set that is rotated about the origin towards the first diagonal of parameter space (c_1, c_2).

With an appropriate choice of the two free parameters, namely,

$$x_{n+1} = 2 x_n y_n - y_n^2 + c_1 - c_2/\sqrt{3},$$
$$y_{n+1} = 2 x_n y_n - x_n^2 + c_1 + c_2/\sqrt{3}, \tag{5}$$

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the Mandelbrot set produced by (4) can be transformed into the original one. The factor $1/\sqrt{3}$ is equal to the ratio between the horizontal and the vertical coordinates of the horizontal tangencies of the cardioid when the origin is the main cusp [5]. These points are also the osculating points of the main side bubbles of the Mandelbrot cardioid [6].

For both (1) and (5) it holds true that the only finite critical point (with zero Jacobian determinant) is the origin.

Figures 1 and 2 show the corresponding two sets. All pictures are plotted in the convention introduced by Mandelbrot [2] and used in [6]: iterate the orbits of the critical point for varying parameters and color points of divergence and convergence differently. (In both Figs. 1 and 2 the convention is used [6] that outside the main region the levels of equal speed of divergence are added in alternating color.) There is no numerical difference between the sets obtained from (1) and (5), not only in Figs. 1 and 2 up to almost every point, but also for larger magnification factors. For example, two close-ups of the region called seahorse valley [6] for both maps show the same striking features (Figs. 3 and 4). The only difference between Figs. 1 and 2 concerns the dynamics of divergence. At low level-line numbers the outermost levels of equal divergence in Fig. 2 oscillate compared to Figure 1. The closer one approaches the main body the smaller the difference becomes. One expects the length and the direction of the 'escape vectors' to differ slightly outside the main body due to the lack of analyticity in (5).

Finally note that (5) clearly violates the Cauchy-Riemann differential equations (3). With

$$f_1 = 2 x y - y^2 + c_1 - c_2/\sqrt{3}$$

and

$$f_2 = 2 x y - x^2 + c_1 + c_2/\sqrt{3},$$

condition (3a) is not fulfilled:

$$\partial f_1(x, y)/\partial x = 2 y \neq \partial f_2(x, y)/\partial y = 2 x.$$

Conclusion

Analyticity seems not to be a necessary condition to get a Mandelbrot-like set. Even 'Julia sets' obtained from (5) are quantitatively equal to those obtained with the same parameter values from (1). (So far only the parameter values displayed on p. 12 of [6] were tested.) Possibly, "generalized Cauchy-Riemann conditions" can be formulated for the system (5) (work in preparation).

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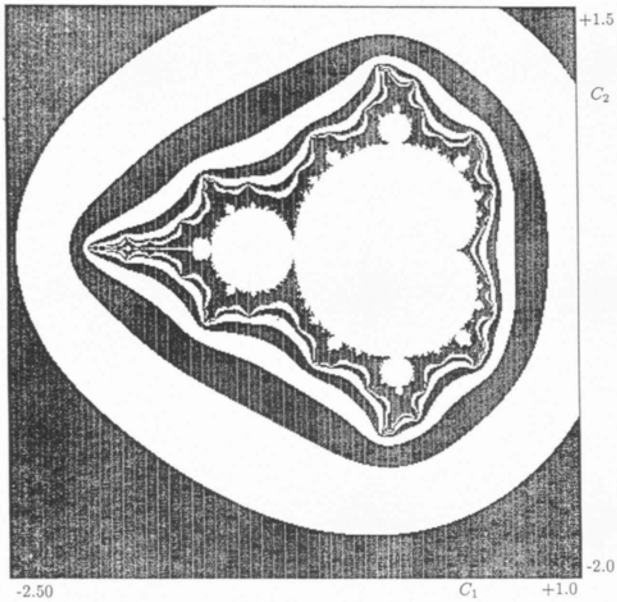

Fig. 1. Mandelbrot set of (2). Window presented: $c_1 = -2.50 \div 1.0$, $c_2 = -2.0 \div 1.5$. 19 digits calculation. Maximum iteration number per point $k = 1000$. Applied criterion for divergence: $x^2 + y^2 > 100$. The outside level lines start for $k < 10$ iterations.

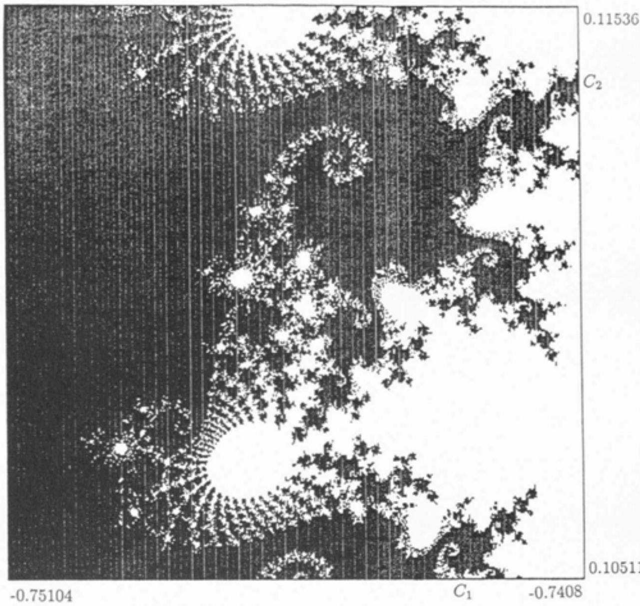

Fig. 3. Close-up of Mandelbrot set of (2). Window presented: $c_1 = -0.75104 \div -0.7408$, $c_2 = 0.10511 \div 0.11536$ [7]. (Compare [6], map 36.) Maximum iteration number per point $k = 200$. Other conditions as in Figure 1.

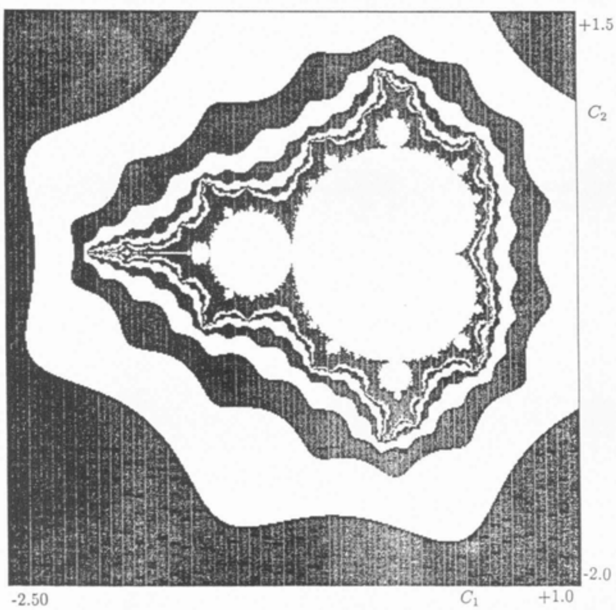

Fig. 2. Mandelbrot set of (5). Window and conditions as in Figure 1.

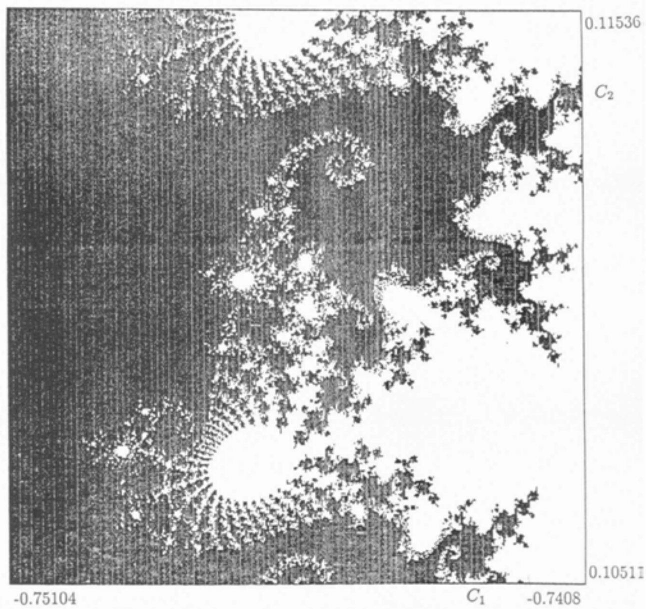

Fig. 4. Close-up of Mandelbrot set of (5). Window and conditions as in Figure 3.